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# An Exponential Hesitant Fuzzy Entropy Measure

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**Abstract**—*Hesitant fuzzy set (HFS) theory provides a sufficiently good tool to model various real-life problems which require human reasoning. The hesitant fuzzy entropy is used for determining the objective weights of attributes in multiple-attribute group decision-making problems. Many entropy measures have been developed by various researchers in the hesitant fuzzy settings. But none of these considered as a panacea, therefore, a new measure is always desirable. In this work, we proposed an exponential hesitant fuzzy entropy measure and derive a characterization result for hesitant fuzzy entropy that enables us to propose a class of entropy measures. Also with the help of comparative study we observe that our proposed exponential hesitant fuzzy entropy measure is effective like other hesitant fuzzy entropy measures while dealing with linguistic hedges. The similarity and dissimilarity measures in the hesitant fuzzy environment also seem to provide new hesitant fuzzy entropies and vice-versa. In this regard, we also investigate the connection of the dissimilarity measures with entropy in the hesitant fuzzy routine.*

**Index Terms**—*fuzzy set, hesitant fuzzy set, hesitant fuzzy entropy, dissimilarity measure, linguistic hedges.*

## 1. INTRODUCTION

In our real-life endeavors, nothing is completely true or false. Zadeh [15] proposed the fuzzy set theory to overcome some constraints of Aristotelean binary logic. The fuzzy logic enables the researcher to model the problems which require human thinking. Many extensions of the fuzzy theory have been provided by the researchers; prominent among them are type-2 fuzzy sets (Dubious and Prade [2], Miyamoto [6]), intuitionistic fuzzy sets (Atanassov [1]), fuzzy multisets (Yager [14]) and hesitant fuzzy sets (Torra [8], Xu and Zhang [12]), etc. As a matter of fact, the human behavior is not deterministic. There is always an element of subjectivity. Torra [8] by proposing the HFS represented the subjectivity of human behavior to assign grade(s) to the impartial information. In hesitant fuzzy settings a member of universal set assumes more than one grading. Some recent information theoretic investigations in the hesitant fuzzy framework have been done in Xu and Zhang [12], Liao and Xu [5], Xu and Xia [9], Xu and Xia [10], Farhadinia [3], etc. Some prominent investigations of HFS in decision-making problems are given in Xu and Xia [11] and Hu et al. [4].

In the hesitant fuzzy information theoretic studies, we have observed some research gaps as follows:

- No general method is obtained to derive a class of hesitant fuzzy entropy measure.
- A connection of hesitant fuzzy entropy with hesitant fuzzy dissimilarity measure is not given in detail.

These research gaps motivated us to consider this work. The main contribution of this work is:

- We derive a characterization result for hesitant fuzzy entropy.
- We propose a new exponential hesitant fuzzy entropy and show its effectiveness like other existing hesitant fuzzy entropies through comparative study while dealing with linguistic hedges.
- We discuss the relation of hesitant fuzzy entropy with hesitant fuzzy dissimilarity measure.

The remainder of the paper is organized as follows:

Section 2 presents some preliminary definitions. In Section 3, we propose an exponential hesitant fuzzy entropy measure and prove some of its properties. Also, a characterization result for hesitant fuzzy entropy is introduced in this section. We also discuss the relation of hesitant fuzzy entropy with hesitant fuzzy dissimilarity measure. Through comparative study, we show the effectiveness of the proposed hesitant fuzzy entropy measure like other existing hesitant fuzzy entropies in Section 4. Section 5 presents the conclusion.

## 2. PRELIMINARIES

Torra [8] introduced the notion of hesitant fuzzy set which is given as follows.

**Definition 1** A HFS $N$ for a fixed set $X$ is expressed as a function $\Theta_N(x)$ that when applied to $X$ returns a subset of [0, 1]. It can be mathematically given as

$$N = \{< x; \Theta_N(x) > | x \in X\}.$$

Here $\Theta_N(x)$ is the possible grade ship value of the element $x \in X$ to the set $N$. Xia and Xu [13] called $\Theta_N(x)$ as an hesitant fuzzy element (HFE) and $N$ the set of all HFEs.

Torra [8] introduced some operations for given three HFEs $\Theta, \Theta_1, \Theta_2$ which are defined as follows:

a) $\Theta^c(x) = \bigcup_{\delta \in \Theta(x)} \{1-\delta\}$,

b) $(\Theta_1 \cup \Theta_2)(x) = \bigcup_{\delta_1 \in \Theta_1(x), \delta_2 \in \Theta_2(x)} \max\{\delta_1, \delta_2\}$,

c) $(\Theta_1 \cap \Theta_2)(x) = \bigcup_{\delta_1 \in \Theta_1(x), \delta_2 \in \Theta_2(x)} \min\{\delta_1, \delta_2\}$.

Xu and Xia [11] provided the following axiomatic definition of similarity measure for hesitant fuzzy sets.

**Definition 2** The similarity measure for two hesitant fuzzy elements $\Theta$ and $\Gamma$ should satisfy the following four properties:

**(S1)** $s(\Theta,\Gamma)=0$ if and only if $\Theta=0, \Gamma=1$ or $\Theta=1, \Gamma=0$;

**(S2)** $s(\Theta,\Gamma)=1$ if and only if $\Theta_{\sigma(i)}=\Gamma_{\sigma(i)}$, $i=1,2,...l$;

**(S3)** $s(\Theta,\delta)\le s(\Theta,\Gamma), s(\Theta,\delta)\le s(\Gamma,\delta)$ if $\Theta_{\sigma(i)} \le \Gamma_{\sigma(i)} \le \delta_{\sigma(i)}$ or $\Theta_{\sigma(i)} \ge \Gamma_{\sigma(i)} \ge \delta_{\sigma(i)}$, $i=1,2,...l$;

**(S4)** $s(\Theta,\Gamma)=s(\Gamma,\Theta)$.

**Definition 3** The dissimilarity measure for two hesitant fuzzy elements $\Theta$ and $\Gamma$ should satisfy the following four properties:

**(D1)** $d(\Theta,\Gamma)=1$ if and only if $\Theta=0, \Gamma=1$ or $\Theta=1, \Gamma=0$;

**(D2)** $d(\Theta,\Gamma)=0$ if and only if $\Theta_{\sigma(i)}=\Gamma_{\sigma(i)}$, $i=1,2,...l$;

**(D3)** $d(\Theta,\delta)\ge d(\Theta,\Gamma), d(\Theta,\delta)\ge d(\Gamma,\delta)$ if $\Theta_{\sigma(i)} \le \Gamma_{\sigma(i)} \le \delta_{\sigma(i)}$ or $\Theta_{\sigma(i)} \ge \Gamma_{\sigma(i)} \ge \delta_{\sigma(i)}$, $i=1,2,...l$;

**(D4)** $d(\Theta,\Gamma)=d(\Gamma,\Theta)$.

Xu and Xia [11] provided the following axiomatic definition of hesitant fuzzy entropy.

**Definition 4** An entropy of HFE $\Theta$ is a real-valued function $E: H \to [0,1]$ satisfying the following axiomatic requirements:

**(E1)** $E(\Theta)=0$ if and only if $\Theta=0$ or $\Theta=1$,

**(E2)** $E(\Theta)=1$ if and only if $\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}=1$, for $i=1,2,...,l$,

**(E3)** $E(\Theta)\le E(\Gamma)$ if $\Theta_{\sigma(i)} \le \Gamma_{\sigma(i)}$ for $\Gamma_{\sigma(i)}+\Gamma_{\sigma(l-i+1)} \le 1$ or $\Gamma_{\sigma(i)}+\Gamma_{\sigma(l-i+1)} \ge 1$, $i=1,2,...,l$,

**(E4)** $E(\Theta)=E(\Theta^c)$.

Singh and Lalotra [7] defined the modifier of a hesitant fuzzy set as follows:

**Definition 6** Consider the reference set X.

Let $f_N = \{(x, f_{N\sigma(1)}(x), f_{N\sigma(2)}(x), f_{N\sigma(3)}(x)) | x \in X\}$ be a hesitant fuzzy element in $X$. Then the modifier for the hesitant fuzzy element $f_N$ is given by $f_N^\gamma = \{(x, (f_{N\sigma(1)}(x))^\gamma, (f_{N\sigma(2)}(x))^\gamma, (f_{N\sigma(3)}(x))^\gamma) | x \in X\}$,

where Very ( $f_N$ ) = CON ( $f_N$ ) = $f_N^\gamma$ *for* $\gamma > 1$ and More or less ( $f_N$ ) = DIL( $f_N$ ) = $f_N^\gamma$ *for* $\gamma < 1$.

Let $N$ be a HFS then we can generate the following HFSs using the above operations which are given below:

$N^{\frac{1}{2}}$ is considered as 'More or Less Large', $N^2$ is considered as 'Very Large', $N^3$ is considered as 'Quite Very Large' and $N^4$ is considered as 'Very Very Large'.

In the next section, we introduce an exponential entropy measure for hesitant fuzzy sets.

## 3. EXPONENTIAL HESITANT FUZZY ENTROPY

We propose the following exponential entropy measure for hesitant fuzzy sets.

$$E^{HF}(\Theta) = \frac{1}{l\sqrt{e}-1} \sum_{i=1}^{l} \begin{bmatrix} \left(\frac{\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}}{2}\right) \\ \times e^{\left(1-\left(\frac{\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}}{2}\right)\right)} \\ +\left(\frac{2-\Theta_{\sigma(i)}-\Theta_{\sigma(l-i+1)}}{2}\right) \\ \times e^{\left(1-\left(\frac{2-\Theta_{\sigma(i)}-\Theta_{\sigma(l-i+1)}}{2}\right)\right)} - 1 \end{bmatrix}. \quad (3.1)$$

**Theorem 1** $E^{HF}(\Theta)$ is a valid hesitant fuzzy entropy measure.

**Proof (E1)** First suppose $\Theta = 0\, or\, \Theta = 1$.

Then for $\Theta = 0$, we have

$E^{HF}(\Theta)=0.$

Also, for $\Theta = 1$, we haves

$E^{HF}(\Theta)=0.$

$\therefore E^{HF}(\Theta)=0$ when $\Theta = 0\, or\, \Theta = 1$.

Conversely suppose $E^{HF}(\Theta)=0$

$$\Rightarrow \frac{1}{l\sqrt{e}-1}\sum_{i=1}^{l}\left[\begin{array}{l}\left(\frac{\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}}{2}\right)\\ \times e^{\left(1-\left(\frac{\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}}{2}\right)\right)}\\ +\left(\frac{2-\Theta_{\sigma(i)}-\Theta_{\sigma(l-i+1)}}{2}\right)\\ \times e^{\left(1-\left(\frac{2-\Theta_{\sigma(i)}-\Theta_{\sigma(l-i+1)}}{2}\right)\right)}-1\end{array}\right]=0.$$

This is possible if $\Theta = 0\, or\, \Theta = 1$.

$\therefore\ E^{HF}(\Theta)=0$ iff $\Theta = 0\, or\, \Theta = 1$.

**(E2)** First suppose $\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}=1$. Then,

$E^{HF}(\Theta)$=1.

$\therefore\ E^{HF}(\Theta)=1$ when $\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}=1$.

Conversely suppose $E^{HF}(\Theta)=1$. Then $E^{HF}(\Theta)=1$

$$\Rightarrow \frac{1}{l\sqrt{e}-1}\sum_{i=1}^{l}\left[\begin{array}{l}\left(\frac{\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}}{2}\right)\\ \times e^{\left(1-\left(\frac{\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}}{2}\right)\right)}\\ +\left(\frac{2-\Theta_{\sigma(i)}-\Theta_{\sigma(l-i+1)}}{2}\right)\\ \times e^{\left(1-\left(\frac{2-\Theta_{\sigma(i)}-\Theta_{\sigma(l-i+1)}}{2}\right)\right)}-1\end{array}\right]=1.$$

This is possible if $\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}=1$.

$\therefore\ E^{HF}(\Theta)=1\ iff\ \Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}=1$.

**(E3)** Let $\Theta_{\sigma(i)}\le\Gamma_{\sigma(i)}$

$\Rightarrow \Theta_{\sigma(l-i+1)}\le\Gamma_{\sigma(l-i+1)}$.

$\therefore\ \Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}\le\Gamma_{\sigma(i)}+\Gamma_{\sigma(l-i+1)}$.

Now, when $\Theta_{\sigma(i)}\le\Gamma_{\sigma(i)}$. Then

$$E^{HF}(\Theta)=\frac{1}{l\sqrt{e}-1}\sum_{i=1}^{l}\left[\frac{x_i}{2}e^{1-\frac{x_i}{2}}+\left(1-\frac{x_i}{2}\right)e^{\frac{x_i}{2}}-1\right],$$

where, $x_i=\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}$.

Now,

$$\frac{\partial E^{HF}}{\partial(x_i)}=\frac{1}{l\sqrt{e}-1}\sum_{i=1}^{l}\left[\frac{1}{2}\left(1-\frac{x_i}{2}\right)e^{1-\frac{x_i}{2}}-\frac{x_i}{4}e^{\frac{x_i}{2}}\right]\ge 0.$$

$\therefore E^{HF}(\Theta)$ is increasing function.

$\therefore\ \Theta_{\sigma(i)}\le\Gamma_{\sigma(i)} \Rightarrow E^{HF}(\Theta)\le E^{HF}(\Gamma)$.

Also, for $\Theta_{\sigma(i)}\ge\Gamma_{\sigma(i)}$, $\frac{\partial E^{HF}}{\partial(x_i)}\le 0$.

$\therefore\ E^{HF}(\Theta)$ is decreasing function.

$\therefore\ \Theta_{\sigma(i)}\ge\Gamma_{\sigma(i)} \Rightarrow E^{HF}(\Theta)\le E^{HF}(\Gamma)$.

Therefore, $E_{\alpha}^{HF}(\Theta)\le E_{\alpha}^{HF}(\Gamma), if\ \Theta_{\sigma(i)}\le\Gamma_{\sigma(i)}$ for $\Gamma_{\sigma(i)}+\Gamma_{\sigma(l_\Theta-i+1)}\le 1\ or\ \Theta_{\sigma(i)}\ge\Gamma_{\sigma(i)}$ for $\Gamma_{\sigma(i)}+\Gamma_{\sigma(l-i+1)}\ge 1,\ i=1,2,...l$.

**(E4)** Clearly, $E^{HF}(\Theta)=E^{HF}(\Theta^c)$.

Hence, $E^{HF}(\Theta)$ is a valid hesitant fuzzy entropy measure.

**Theorem 2** Let $E^{HF}(\Theta_1)$ and $E^{HF}(\Theta_2)$ be hesitant fuzzy entropy measures of HFEs $\Theta_1$ and $\Theta_2$ respectively. Then $E^{HF}(\Theta_1\cup\Theta_2)+E^{HF}(\Theta_1\cap\Theta_2)=E^{HF}(\Theta_1)+E^{HF}(\Theta_2)$.

Now, to obtain a general framework for obtaining new entropy measures in hesitant fuzzy environment we prove characterization result.

**Theorem3** Let $F:[0,1]\to[0,1]$ be a mapping. The function $E_F^{HF}:HF(X)\to[0,1]$ defined by

$E_F^{HF}(\Theta)=\frac{1}{l}\sum_{i=1}^{n}F\left(\frac{\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}}{2}\right)$, satisfies axioms (E 1) - (E4) if

(i) $E_F^{HF}$ satisfy (E1) if and only if $F(0)=0=F(1)$, $F(x)\ne 0\ \forall x\in(0,1)$,

(ii) $E_F^{HF}$ satisfy (E2) if and only if $F(x)<F\left(\frac{1}{2}\right)$, $\forall x\in[0.1]-\left\{\frac{1}{2}\right\}$,

(iii) $E_F^{HF}$ satisfy (E3) if and only if $F(x)$ is increasing on [0, 0.5] and $F(x)$ is decreasing on [0.5, 1],

(iv) $E_F^{HF}$ satisfy (E4) if and only if $F(x)=F(1-x)\ \forall\ x\in[0,1]$.

**Proof** Suppose $E_F^{HF}$ satisfies (E1)-(E4). We have to show that $F$ satisfies (i)-(iv).

(i) Suppose $E_F^{HF}$ satisfy (E1). We have to show that $F(0)=0=F(1)$. We have

$$E_F^{HF}(\Theta)=\frac{1}{l}\sum_{i=1}^{l}F\left(\frac{\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}}{2}\right)$$

Now, from (E1), we have

$E_F^{HF}(A)$=0 if and only if $\Theta=0$ or 1

i.e., $\frac{1}{l}\sum_{i=1}^{l}F\left(\frac{\Theta_{\sigma(i)}+\Theta_{\sigma(l-i+1)}}{2}\right)=0$ if and only if $\Theta=0$ or 1.

i.e., $\frac{1}{l}\sum_{i=1}^{l} F(0) = \frac{1}{l}\sum_{i=1}^{l} F(1) = 0$ if and only if $F(0) = F(1) = 0$.

i.e., $F(0) = F(1) = 0$.

Converse can be obtained by just reversing the above step.

This proves (i).

(ii) Suppose $E_F^{HF}$ satisfy (E2) i.e. $E_F^{HF}(\Theta) = 1$ if and only if $\Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)} = 1$. We have to show that

$$F(x) < F\left(\frac{1}{2}\right),\ \forall x \in [0.1] - \left\{\frac{1}{2}\right\}.$$

Now, $E_F^{HF}(\Theta) = 1$ if and only if $\Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)} = 1$

$\Rightarrow \frac{1}{l}\sum_{i=1}^{l} F\left(\frac{\Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)}}{2}\right) = 1$ if and only if $\Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)} = 1$

if and only if $\frac{1}{l}\sum_{i=1}^{l} F\left(\frac{1}{2}\right) = 1$ if and only if $F\left(\frac{1}{2}\right) = 1$

if and only if $F(x) < F\left(\frac{1}{2}\right),\ \forall x \in [0,1] - \left\{\frac{1}{2}\right\}$.

Converse can be obtained by just reversing the above step.

This proves (ii).

(iii) Now suppose $E_F^{HF}$ satisfy (E3). We have to show that $F(x)$ is increasing on [0, 0.5] and $F(x)$ is decreasing on [0.5, 1).

Now, if possible suppose $F(x)$ is decreasing on [0, 0.5].

Now, from property (E3) we have

$E_F^{HF}(\Theta) \le E_F^{HF}(\Gamma)$ when $\Theta_{\sigma(i)} \le \Gamma_{\sigma(i)}$ for

$$\Gamma_{\sigma(i)} + \Gamma_{\sigma(l-i+1)} \le 1 \ . \tag{3.2}$$

By our assumption $F(x)$ is decreasing on [0, 0.5]

$$F\left(\frac{\Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)}}{2}\right) \ge F\left(\frac{\Gamma_{\sigma(i)} + \Gamma_{\sigma(l-i+1)}}{2}\right)$$

$$\Rightarrow \frac{1}{l}\sum_{i=1}^{l} F\left(\frac{\Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)}}{2}\right) \ge \frac{1}{l}\sum_{i=1}^{l} F\left(\frac{\Gamma_{\sigma(i)} + \Gamma_{\sigma(l-i+1)}}{2}\right)$$

$\Rightarrow E_F^{HF}(\Theta) \ge E_F^{HF}(\Gamma)$

which contradicts (3.2).

Hence our supposition is wrong.

$\therefore$ $F(x)$ is increasing on [0, 0.5]. Similarly, we show that $F(x)$ is decreasing on [0.5, 1].

Converse can be obtained by just reversing the above step.

(iv) Suppose, $E_F^{HF}$ satisfies (E4). Then

$E_F^{HF}(\Theta) = E_F^{HF}(\Theta^c)$.

We have to show that $F(x) = F(1-x),\ \forall x \in [0,1]$.

If possible suppose $F(x) \ne F(1-x)$. Then without loss generality $F(x) > F(1-x)$

$$\Rightarrow F\left(\frac{\Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)}}{2}\right) > F\left(1 - \frac{\Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)}}{2}\right)$$

$$\Rightarrow \frac{1}{l}\sum_{1=1}^{l} F\left(\frac{\Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)}}{2}\right) > \frac{1}{l}\sum_{1=1}^{l} F\left(\frac{2 - \Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)}}{2}\right)$$

$\Rightarrow E_F^{HF}(\Theta) > E_F^{HF}(\Theta^c)$.

This is a contradiction to (E4).

Therefore, $F(x) = F(1-x), \forall x \in [0,1]$.

Conversely, suppose $F(x) = F(1-x), \forall x \in [0,1]$

$$\Rightarrow F\left(\frac{\Theta_{\sigma(i)} + \Theta_{\sigma(l_{\Theta-i+1})}}{2}\right) = F\left(1 - \frac{\Theta_{\sigma(i)} + \Theta_{\sigma(l_{\Theta-i+1})}}{2}\right)$$

$$\Rightarrow \frac{1}{l}\sum_{1=1}^{l} F\left(\frac{\Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)}}{2}\right) = \frac{1}{l}\sum_{1=1}^{l} F\left(\frac{2 - \Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)}}{2}\right)$$

$\Rightarrow E_F^{HF}(\Theta) = E_F^{HF}(\Theta^c)$.

This proves (iv).

In view of Theorem 3, we can state the following characterization theorem.

**Theorem 4** Let $X = \{x_1, x_2, ..., x_n\}$ and $E_F^{HF}$ : *F(X)* → [0,1] be $E_F^{HF}(\Theta) = \frac{1}{l}\sum_{i=1}^{l} F\left(\frac{\Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)}}{2}\right)$. Then $E_F^{HF}(\Theta)$ satisfies E1-E4 if and only if $E_F^{HF}$ for some function *F*: [0, 1]→[0, 1] satisfies (i)-(iv) (as given in Theorem 3).

**Theorem 5** Let $\Theta$ be an HFE, then $1 - d(\Theta, \Theta^c)$ is an entropy measure of $\Theta$, where *d* is the hesitant fuzzy dissimilarity measure.

**Proof (E1)** Take $1 - d(\Theta, \Theta^c) = E^{HF}(\Theta)$.

We have, $1 - d(\Theta, \Theta^c) = 0$ if and only if $d(\Theta, \Theta^c) = 1$

if and only if $\Theta = 0$ and $\Theta^c = 1$ or $\Theta = 1$ and $\Theta^c = 0$; (Using D1)

i.e., $E^{HF}(\Theta) = 1$ if and only if $\Theta = 0$ or $\Theta = 1$.

This proves E1.

**(E2)** $1 - d(\Theta, \Theta^c) = 1$ if and only if $d(\Theta, \Theta^c) = 0$

if and only if $\Theta = \Theta^c$ (Using D4)

i.e., $E^{HF}(\Theta) = 0$ if and only if $\Theta_{\sigma(i)} + \Theta_{\sigma(l-i+1)} = 1$ for *i*= 1, 2, . . . , *l;*

This proves E2.

**(E3)** Suppose that $\Theta_{\sigma(i)} \le \Gamma_{\sigma(i)}$, for $\Gamma_{\sigma(i)} + \Gamma_{\sigma(l-i+1)} \le 1$, then $\Theta_{\sigma(i)} \le \Gamma_{\sigma(i)} \le 1 - \Gamma_{\sigma(l-i+1)} \le 1 - \Theta_{\sigma(l-i+1)}$.

Therefore, by the axiom D3 of the dissimilarity measure of HFE, we have

$d(\Theta, \Theta^c) \ge d(\Gamma, \Theta^c) \ge d(\Gamma, \Gamma^c)$ .

This implies

$1-d(\Theta,\Theta^c) \le 1-d(\Gamma,\Theta^c) \le 1-d(\Gamma,\Gamma^c)$

i.e., $1-d(\Theta,\Theta^c) \ge 1-d(\Gamma,\Gamma^c)$

i.e., $E^{HF}(\Theta) \le E^{HF}(\Gamma)$ if $\Theta_{\sigma(i)} \le \Gamma_{\sigma(i)}$, for $\Gamma_{\sigma(i)} + \Gamma_{\sigma(l-i+1)} \le 1$.

Similarly, we can prove it when $\Theta_{\sigma(i)} \ge \Gamma_{\sigma(i)} \ge 1-\Gamma_{\sigma(l-i+i)} \ge 1-\Theta_{\sigma(l-i+i)}$, $i$= 1, 2, . . . , $l$.

Thus axiom E3 follows.

**(E4)** Using axiom D1, we have

$d(\Theta,\Theta^c) = d(\Theta^c,\Theta)$. This gives $1-d(\Theta,\Theta^c) = 1-d(\Theta^c,\Theta)$

i.e., $E^{HF}(\Theta) = E^{HF}(\Theta^c)$.

This proves E4.

Therefore, $1-d(\Theta,\Theta^c) = E_\alpha{}^{HF}(\Theta)$ is a valid entropy measure of HFE $\Theta$.

Next section provides the comparative study of the proposed hesitant fuzzy entropy measure.

## 4. COMPARATIVE STUDY

Let $\Theta_N = \{(x, \Theta_{N\sigma(1)}(x), \Theta_{N\sigma(2)}(x), \Theta_{N\sigma(3)}(x)) \mid x \in X\}$ be a hesitant fuzzy element in $X$. Then the modifier for the hesitant fuzzy element $\Theta_N$ is given by

$$\Theta_N^\gamma = \left\{ \begin{array}{l} (x, (\Theta_{N\sigma(1)}(x))^\gamma, (\Theta_{N\sigma(2)}(x))^\gamma, \\ (\Theta_{N\sigma(3)}(x))^\gamma) \mid x \in X \end{array} \right\}., \text{ where}$$

Very ($\Theta_N$) = CON ($\Theta_N$) = $\Theta_N^\gamma$ *for* $\gamma > 1$ and More or less ($\Theta_N$) = DIL($\Theta_N$) = $\Theta_N^\gamma$ *for* $\gamma < 1$.

**Example** In general, if $\Theta_N = \{(\Theta_{N\sigma(1)}, \Theta_{N\sigma(2)}, \Theta_{N\sigma(3)}\}$ be a hesitant fuzzy element in $X$. Then the modifier for the fuzzy element $\Theta_N$ is given by

$$\Theta_N{}^\gamma = \{(\Theta_{N\sigma(1)})^\gamma, (\Theta_{N\sigma(2)})^\gamma, (\Theta_{N\sigma(3)})^\gamma)\}. \quad (4.1)$$

For the sake of clarity in understanding the computations, let $\Theta_N^i$ be a hesitant fuzzy element in the hesitant fuzzy set $N$ corresponding to the reference set $X = \{6,7,8\}$. Numerically, we consider

$$N = \{\Theta_N^1, \Theta_N^2, \Theta_N^3\} = \left\{ \begin{array}{l} (6, \{0.5, 0.4, 0.2\}), (7, \{0.3, 0.3, 0.1\}), \\ (8, \{0.4, 0.3, 0.1\}) \end{array} \right\}.$$

where,

$$\Theta_N^1 = <x, \{\Theta_{N\sigma(1)}^1, \Theta_{N\sigma(2)}^1, \Theta_{N\sigma(3)}^1\}> = <6, \{0.5, 0.4, 0.2\}>,$$

$$\Theta_N^2 = <x, \{\Theta_{N\sigma(1)}^2, \Theta_{N\sigma(2)}^2, \Theta_{N\sigma(3)}^2\}> = <7, \{0.3, 0.3, 0.1\}>,$$

$$\Theta_N^3 = <x, \{\Theta_{N\sigma(1)}^3, \Theta_{N\sigma(2)}^3, \Theta_{N\sigma(3)}^3\}> = <8, \{0.4, 0.3, 0.1\}>.$$

We can generate the following hesitant fuzzy elements by using the operation given in equation (4.1) as follows:

$(\Theta_N^1)^{\frac{1}{2}}, (\Theta_N^1)^2, (\Theta_N^1)^3, (\Theta_N^1)^4; (\Theta_N^2)^{\frac{1}{2}}, (\Theta_N^2)^2, (\Theta_N^2)^3, (\Theta_N^2)^4$

and $(\Theta_N^3)^{\frac{1}{2}}, (\Theta_N^3)^2, (\Theta_N^3)^3, (\Theta_N^3)^4$

By considering the characterization of linguistic variables

$(\Theta_N^i)^{\frac{1}{2}}$ is considered as 'More or Less Large',

$(\Theta_N^i)^2$ is considered as 'Very Large',

$(\Theta_N^i)^3$ is considered as 'Quite Very Large' and

$(\Theta_N^i)^4$ is considered as 'Very Very Large'.

Now, by taking into account the mathematical operations, the entropy of hesitant fuzzy element should follow the following order:

$$E^{HF}(\Theta_N^{\frac{1}{2}}) > E^{HF}(\Theta_N) > E^{HF}(\Theta_N^2) > E^{HF}(\Theta_N^3) > E^{HF}(\Theta_N^4). \quad (4.2)$$

In Table 1, the comparative results for the following hesitant fuzzy entropies and our proposed hesitant fuzzy entropy measure regarding HFE $\Theta_N$ are given.

$$E_1^{HF}(\Theta_N) = -\frac{1}{l\ln 2}\sum_{i=1}^{l}\left[\begin{array}{l}\left(\frac{\Theta_{N\sigma(i)}+\Theta_{N\sigma(l-i+1)}}{2}\right)\ln\left(\frac{\Theta_{N\sigma(i)}+\Theta_{N\sigma(l-i+1)}}{2}\right)+ \\ \left(\frac{2-\Theta_{N\sigma(i)}-\Theta_{N\sigma(l-i+1)}}{2}\right)\ln\left(\frac{2-\Theta_{N\sigma(i)}-\Theta_{N\sigma(l-i+1)}}{2}\right)\end{array}\right];$$

(Xu and Xia [11])

$$E_2^{HF}(\Theta_N) = 1-\frac{1}{l}\sum_{\delta\in\Theta_N}\min_{\delta^c\in\Theta_N^c}\left|\delta-\delta^c\right|;$$ (Hu *et al.* [4])

$$E_3^{HF}(\Theta_N) = 1-\left[\frac{1}{l}\sum_{\delta\in\Theta_N}\min_{\delta^c\in\Theta_N^c}\left|\delta-\delta^c\right|^2\right]^{\frac{1}{2}};$$ (Hu *et al.*[4])

$$E_4^{HF}(\Theta_N) = \frac{1}{l(\sqrt{2}-1)}\sum_{i=1}^{l}\left(\begin{array}{l}\sin\frac{\pi(\Theta_{N\sigma(i)}+\Theta_{N\sigma(l-i+1)})}{4}+ \\ \sin\frac{\pi(2-\Theta_{N\sigma(i)}-\Theta_{N\sigma(l-i+1)})}{4} \\ -1\end{array}\right);$$

(Xu and Xia [11])

$$E_5^{HF}(\Theta_N) = \frac{1}{l(\sqrt{2}-1)}\sum_{i=1}^{l}\left(\begin{array}{l}\cos\frac{\pi(\Theta_{N\sigma(i)}+\Theta_{N\sigma(l-i+1)})}{4}+ \\ \cos\frac{\pi(2-\Theta_{N\sigma(i)}-\Theta_{N\sigma(l-i+1)})}{4} \\ -1\end{array}\right);$$

(Xu and Xia [11])

Now, from Table 1 we observe

$E_1^{HF}(\Theta_N^1)^{\frac{1}{2}} > E_1^{HF}(\Theta_N^1) > E_1^{HF}(\Theta_N^1)^2 > E_1^{HF}(\Theta_N^1)^3 > E_1^{HF}(\Theta_N^1)^4$,

$E_2^{HF}(\Theta_N^1)^{\frac{1}{2}} > E_2^{HF}(\Theta_N^1) > E_2^{HF}(\Theta_N^1)^2 > E_2^{HF}(\Theta_N^1)^3 > E_2^{HF}(\Theta_N^1)^4$,

$E_3^{HF}(\Theta_N^1)^{\frac{1}{2}} > E_3^{HF}(\Theta_N^1) > E_3^{HF}(\Theta_N^1)^2 > E_3^{HF}(\Theta_N^1)^3 > E_3^{HF}(\Theta_N^1)^4$,

$E_4^{HF}(\Theta_N^1)^{\frac{1}{2}} > E_4^{HF}(\Theta_N^1) > E_4^{HF}(\Theta_N^1)^2 > E_4^{HF}(\Theta_N^1)^3 > E_4^{HF}(\Theta_N^1)^4$,

$E_5^{HF}(\Theta_N^1)^{\frac{1}{2}} > E_5^{HF}(\Theta_N^1) > E_5^{HF}(\Theta_N^1)^2 > E_5^{HF}(\Theta_N^1)^3 > E_5^{HF}(\Theta_N^1)^4$, and

$E^{HF}(\Theta_N^1)^{\frac{1}{2}} > E^{HF}(\Theta_N^1) > E^{HF}(\Theta_N^1)^2 > E^{HF}K(\Theta_N^1)^3 > E^{HF}(\Theta_N^1)^4$.

**Table 1 Comparative results of the exponential hesitant fuzzy entropy measure with different hesitant fuzzy entropies for $\Theta_N^1$.**

| HFEs | $E_1^{HF}(\Theta_N^1)$ | $E_2^{HF}(\Theta_N^1)$ | $E_3^{HF}(\Theta_N^1)$ | $E_4^{HF}(\Theta_N^1)$ | $E_5^{HF}(\Theta_N^1)$ | $E^{HF}(\Theta_N^1)$ |
|---|---|---|---|---|---|---|
| $(\Theta_N^1)^{\frac{1}{2}}$ | 0.896 | 0.8902 | 0.4668 | 0.9590 | 0.9590 | 0.9627 |
| $\Theta_N^1$ | 0.8667 | 0.8174 | 0.4547 | 0.9231 | 0.9231 | 0.9299 |
| $(\Theta_N^1)^2$ | 0.4 | 0.3939 | 0.2929 | 0.4967 | 0.4967 | 0.5216 |
| $(\Theta_N^1)^3$ | 0.191 | 0.1896 | 0.1679 | 0.2357 | 0.2357 | 0.2542 |
| $(\Theta_N^1)^4$ | 0.0924 | 0.0921 | 0.0930 | 0.1107 | 0.1107 | 0.1209 |

From above table, it is clear that for HFE $\Theta_N^1$ the hesitant fuzzy entropy measures $E_1^{HF}, E_2^{HF}, E_3^{HF}, E_4^{HF}$ and $E_5^{HF}$ are satisfying the requirement (4.2). Also, our proposed hesitant fuzzy entropy measure satisfies the requirement (4.2). Similar result is obtained for HFE $\Theta_N^2$ and $\Theta_N^3$. Thus like other entropy measures our proposed entropy measure is also effective while dealing with linguistic hedges.

## 5. CONCLUSION

In this work, we have introduced an exponential hesitant fuzzy entropy measure and with the help of comparative study, we show that it is also effective like other hesitant fuzzy entropy measures while dealing with linguistic hedges. We also discuss the relation of hesitant fuzzy entropy with hesitant fuzzy dissimilarity measure.

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